THOUGHTS ON OPTIMAL CONTROL

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ABSTRACT
The purpose of this talk is to outline a, seemingly new, approach to a wide variety of optimal control problems for linear, causal, time-invariant systems. This approach has the advantages of not being restricted to finite-dimensional systems, and has extensions to optimization problems for various classes of transfer functions, including positive real and bounded real functions.

Control theory has developed over the years into a very broad subject, making it difficult to get a good grasp on the various aspects of the subject and the way they are related. Even restricting ourselves to linear, time-invariant, systems, this difficulty is enhanced by the wide choice of system descriptions. We can choose to use external, that is input/output descriptions, or internal descriptions, namely models that explain the external behavior of the given system. Models are far from unique and state space models are but one of many. In fact, even if state space is the form we may prefer for computational purposes, it may not be the best representation for the analysis, and solution, of most control problems. There is another choice to be made due to the possibility of passing on, using various transforms, from the time domain to the frequency domain. In many cases, the frequency domain provides a setting with a richer functional structure that facilitates the solution of the problems of interest.

Apart from the setting, there is a wide variety of control problems to be considered. These include robust stabilization, model reduction, optimal regulator and estimation problems. In order to gain a good understanding of the subject, it is not enough to find a solution to any particular problem. It is of utmost importance to also clarify the relations between different aspects of the theory. Thus, whenever possible, we indicate different approaches to a particular result.

The choice we made is to work mostly in the frequency domain setting, in particular in the use of vectorial Hardy spaces $H^2_\pm$ as signal spaces and co-invariant subspaces as state spaces for the system. This choice has the additional advantage of being able to use the algebraic theory, emphasizing the polynomial module structure, as a guide.

Although the technicalities of polynomial model based system theory for discrete time linear systems over an arbitrary field are vastly different from the Hardy space based theory for some classes of continuous-time systems, there are strong algebraic similarities. These, with the help of heavy analytic tools, can be used to extend the algebraic approach to a wide variety of optimal control and estimation problems for several classes of, not necessarily rational, analytic functions. Due to the underlying Hilbert space structure of Hardy spaces, the treatment of optimal control problems are greatly simplified. As we shall try to show, this has the potential of leading to a grand unification of optimal control theory. Thus, (doubly) coprime factorizations over $H^\infty_\pm$ play a central role. Although many of the theorems we use are true in appropriate infinite dimensional setting, presently, we shall deal mostly with the finite dimensional case. Other than studying infinite dimensional systems, this opens up the possibility of extending the methods to other settings as, for example to special classes of systems (positive real, bounded real). Another challenging direction for future research is to extend optimal control theory to deal with complexity, that is, to networks of systems, using local optimality results for the nodes as well as the interconnection data.

Here, in telegraphic style, is an outline of the suggested approach to optimal control theory for stable systems. It is based on [7].

- Describe the optimization problems in the time domain from the input/output point of view. Choose the signal spaces to be $L^2_{(-\infty,\infty)}$ spaces. Introduce the left and right translation groups. Describe the input/output map in terms of a convolution integral with an appropriate kernel. Characterize causality and boundedness.
- Use the Fourier-Plancherel transform, and the Paley-Wiener theorems, to reformulate the setting to that of the Hardy spaces $H^2_\pm$ setting. Introduce in the Hardy spaces the $H^\infty_\pm$-module structure.
- Discuss stability, transfer functions. Identify the restricted input/output map with a Hankel operator. Characterize Hankel operators as H^∞_--module homomorphisms with respect to the H^∞_--module structures of the Hardy spaces $H^2_\pm$.
- Use the Beurling-Lax characterizations of invariant subspaces, See [1, 10]. Relate the kernel and image of the Hankel operator to the Douglas-Shapiro-Shields factorizations, that is coprime factorizations over H^∞_-, see [2, 4].
- Discuss how inner functions are derived through spectral factorizations, or alternatively, by solving a Lyapunov equation or, alternatively, a homogeneous Ricatti equations.
- Use the Kalman approach to realizations as factorizations to identify the restricted Hankel operator, that is the map from the orthogonal complement of the kernel to the range, as a reachability operator. Both these subspaces are called model spaces and play a central role as state spaces.
- Explain the connection between Hankel operators and intertwining maps, that is $H^\infty_\pm$-homomorphisms, between model spaces.
- Explain how an $H^\infty_\pm$-isomorphism can be inverted by solving a Bezout equation, as in [3], or, even better, by embedding an intertwining relation in a doubly coprime factorization.
- Apply this invertibility procedure to the solution of optimal control problems.
- State the solution in terms of a state space realization.

So far we outlined the frequency domain solution to the optimal control problem for the case of a stable, H^∞_+ transfer function.. This can be taken as a intermediate step towards the analysis of the general, not necessarily stable, case. We present the basic ideas in the same condensed style as before. Some of the ideas and results presented owe much to a cooperation with Raimund Ober in the early 1990s and a long term one with Uwe Helmke, culminating in [8].

- Given the strictly proper transfer function $G(s)$, construct a normalized coprime factorizations of the form
$$G = N_r M_r^{-1} = M_\ell^{-1} N_\ell, \tag{1}$$

with all factors in H_+^∞ and the normalization conditions

$$\begin{pmatrix} M_r^* & N_r^* \\ -N_\ell & M_\ell \end{pmatrix} \begin{pmatrix} M_r & -N_\ell^* \\ N_r & M_\ell^* \end{pmatrix} = \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix}$$

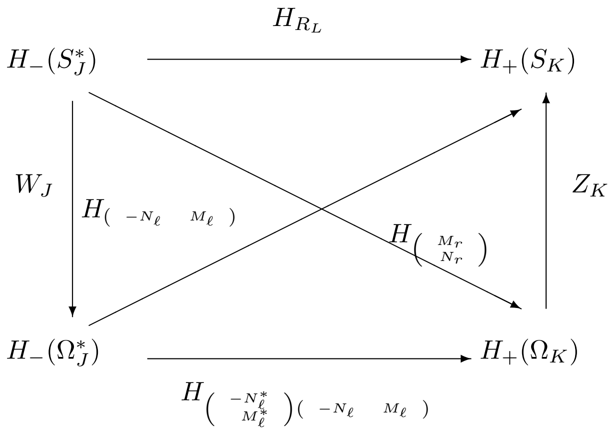
satisfied.

- Derive state space representatins for all the the factors, see [9].
- Derive stabilizing controllers, having the coprime factorization representations $K = U_r V_r^{-1} = V_\ell^{-1} U_\ell$ by solving the Bezout equations $V_\ell M_r + U_\ell N_r = I$ and $M_\ell V_r + N_\ell U_r = I$. Embed in a doubly coprime factorization
$$\begin{pmatrix} V_\ell & U_\ell \\ -N_\ell & M_\ell \end{pmatrix} \begin{pmatrix} M_r & -U_r \\ N_r & V_r \end{pmatrix} = \begin{pmatrix} I & U_\ell V_r - V_\ell U_r \\ 0 & I \end{pmatrix},$$
- Show the existence of a unique, stabilizing, controller for which the **characteristic function** R_L, defined by
$$R_L := U_r^* M_r - V_r^* N_r = M_\ell U_\ell^* - N_\ell V_\ell^*$$
is in H_+^∞ and has the DSS factorization over H_-^∞, given by
$$R_L = \Phi_J^* S_J = S_K \Phi_K^*.$$
For more on characteristic functions, see [5]
- Show that
$$\begin{aligned} \begin{pmatrix} J_1 \\ J_2 \end{pmatrix} &:= \begin{pmatrix} -N_\ell^* \\ M_\ell^* \end{pmatrix} S_K \\ \begin{pmatrix} K_1 & K_2 \end{pmatrix} &:= S_J \begin{pmatrix} M_r^* & N_r^* \end{pmatrix}, \end{aligned}$$
with $J_i, K_i \in H_+^\infty$.
- Show that
$$\begin{aligned} &\operatorname{Ker} H_{\begin{pmatrix} -N_\ell & M_\ell \end{pmatrix}} = \Omega_J^* H_-^2, \\ &\operatorname{Im} H_{\begin{pmatrix} -N_\ell & M_\ell \end{pmatrix}} = H_+(S_K) = \{S_K H_+^2\}^\perp. \\ &\operatorname{Ker} H_{\begin{pmatrix} M_r \\ N_r \end{pmatrix}} = S_J^* H_-^2, \\ &\operatorname{Im} H_{\begin{pmatrix} M_r \\ N_r \end{pmatrix}} = H_+(\Omega_K) = \{\Omega_K H_+^2\}^\perp, \end{aligned}$$
where the inner functions are given by
$$\begin{aligned} \Omega_J &= \begin{pmatrix} -N_\ell & M_\ell \\ K_1 & K_2 \end{pmatrix} \\ \Omega_K &= \begin{pmatrix} M_r & J_1 \\ N_r & J_2 \end{pmatrix}. \end{aligned}$$

- Show that all the maps defined in the following table are H_-^∞-isomorphisms with respect to the appropriate H_-^∞-module structures.

Map	Intertw. Relation	Hom
$Z_K : H_+(\Omega_K) \longrightarrow H_+(S_K)$ $Z_K \begin{pmatrix} f_1 \\ f_2 \end{pmatrix} = P_+ \begin{pmatrix} -U_r^* & V_r^* \end{pmatrix} \begin{pmatrix} f_1 \\ f_2 \end{pmatrix}$	$\begin{pmatrix} \Phi_K^* & I \end{pmatrix} \begin{pmatrix} M_r^* & N_r^* \\ J_1^* & J_2^* \end{pmatrix} = S_K^* \begin{pmatrix} -U_r^* & V_r^* \end{pmatrix}$	H_-^∞
$Z_K^{-1} : H_+(S_K) \longrightarrow H_+(\Omega_K)$ $Z_K^{-1} f = P_+ \begin{pmatrix} -N_\ell^* \\ M_\ell^* \end{pmatrix} f$	$\begin{pmatrix} M_r^* & N_r^* \\ J_1^* & J_2^* \end{pmatrix} \begin{pmatrix} -N_\ell^* \\ M_\ell^* \end{pmatrix} = \begin{pmatrix} 0 \\ I \end{pmatrix} S_K^*$	H_-^∞
$W_J : H_-(S_J^*) \longrightarrow H_-(\Omega_J^*)$ $W_J h = P_{H_-(\Omega_J^*)} \begin{pmatrix} V_\ell^* \\ U_\ell^* \end{pmatrix} h$	$\begin{pmatrix} V_\ell^* \\ U_\ell^* \end{pmatrix} S_J^* = \begin{pmatrix} -N_\ell^* & K_1^* \\ M_\ell^* & K_2^* \end{pmatrix} \begin{pmatrix} -\Phi_J^* \\ I \end{pmatrix}$	H_-^∞
$W_J^{-1} : H_-(\Omega_J^*) \longrightarrow H_-(S_J^*)$ $W_J^{-1} h = P_{H_-(S_J^*)} \begin{pmatrix} M_r^* & N_r^* \end{pmatrix} \begin{pmatrix} h_1 \\ h_2 \end{pmatrix}$	$\begin{pmatrix} M_r^* & N_r^* \end{pmatrix} \begin{pmatrix} -N_\ell^* & K_1^* \\ M_\ell^* & K_2^* \end{pmatrix} = S_J^* \begin{pmatrix} 0 & I \end{pmatrix}$	H_-^∞
$H_{\begin{pmatrix} -N_\ell & M_\ell \end{pmatrix}} : H_-(\Omega_J^*) \longrightarrow H_+(S_K)$ $H_{\begin{pmatrix} -N_\ell & M_\ell \end{pmatrix}} \begin{pmatrix} h_1 \\ h_2 \end{pmatrix}$ $= P_+ \begin{pmatrix} -N_\ell & M_\ell \end{pmatrix} \begin{pmatrix} h_1 \\ h_2 \end{pmatrix}$	$\begin{pmatrix} I & 0 \end{pmatrix} \begin{pmatrix} -N_\ell & M_\ell \\ K_1 & K_2 \end{pmatrix} = S_K \begin{pmatrix} J_1^* & J_2^* \end{pmatrix}$	H_-^∞
$H_R : H_-(S_J^*) \longrightarrow H_+(S_K)$ $H_R h = P_+ R h$	$\Phi_J^* S_J = S_K \Phi_K^*$	H_-^∞
$H_{\begin{pmatrix} M_r \\ N_r \end{pmatrix}} : H_-(S_J^*) \longrightarrow H_+(\Omega_K)$ $H_{\begin{pmatrix} M_r \\ N_r \end{pmatrix}} h = P_+ \begin{pmatrix} M_r \\ N_r \end{pmatrix} h$	$\begin{pmatrix} K_1^* \\ K_2^* \end{pmatrix} S_J = \begin{pmatrix} M_r & J_1 \\ N_r & J_2 \end{pmatrix} \begin{pmatrix} I \\ 0 \end{pmatrix}$	H_-^∞
$H : H_-(\Omega_J^*) \longrightarrow H_+(\Omega_K)$ $H = H_{\begin{pmatrix} M_r \\ N_r \end{pmatrix}\begin{pmatrix} M_r^* & N_r^* \end{pmatrix}} \begin{pmatrix} h_1 \\ h_2 \end{pmatrix}$		H_-^∞

- Explain how all these maps are associated with appropriate optimal control problems. These maps are strongly interrelated through the following commutative diagram.



- Use this diagram, to obtain related ones by applying the adjoint operation to all maps, or, using doubly coprime factorizations, inverting the maps. For example, problems of robust control turn this way into problems of model reduction. In this connection, see [6]. Most of these connections have not yet been worked out.

As the saying goes "god, (or the devil), lies in the details". It can be easily seen, from glancing at the brief outline, that there is an enormous amount of details needed to tell the full story and that would require a monograph. Whether I can do it myself remains to be seen.

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